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INDEPENDENT BLOCK DOMINATION IN GRAPHS

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ABSTRACT

For any graph G(V, E), a block graph B(G) is a graph whose vertices are corresponding to the blocks of G and two vertices in B(G) are adjacent whenever the corresponding blocks contain a common cut vertex in G. A dominating set $I \subseteq V[B(G)]$ is called an independent block dominating set if induced sub graph $\langle I \rangle$ is independent. The independent block domination number is denoted by $\gamma_{ib}(G)$ and equals to $\min \|I\|$, I is an independent block dominating set of B(G). In this paper many bounds on $\gamma_{ib}(G)$ are established in terms of elements of G.

KEYWORDS: Block Graph, Domination Number, Independent Domination Number, Independent Block Domination Number

Subject Classification Number: AMS - 05C69, 05C70

INTRODUCTION

We consider connected, undirected, finite graphs without loops. We follow the notations and terminology of Harary [2]. Let G = (V, E) be a graph with $|V| = p$ and $|E| = q$. n denotes number of blocks of G. The maximum degree of a vertex in G is $\Delta(G)$. The minimum number of colors in any coloring of a graph G such that no two adjacent vertices have same color is called the chromatic number of G and is denoted by $\chi(G)$. The vertex covering number $\alpha_0(G)$ is minimum cardinality of vertex covering where vertex covering of G is a set of vertices that covers all edges of G. N(v) and N [v] denote the open and closed neighborhoods of a vertex v respectively in G.

A set D of a graph G =(V, E) is a dominating set if every vertex in V – D is adjacent to some vertex in D. The domination number $\gamma(G)$ of G is the minimum cardinality of a minimal dominating set. A dominating set D is called connected dominating set if induced sub graph $\langle D \rangle$ is connected. The connected domination number $\gamma_c(G)$ is the minimum cardinality of a connected dominating set of G. A set I_e of edges in a graph G =(V,E) is called an edge dominating set of G if every edge in E – I_e is adjacent to some edge in I_e. The edge domination number $\gamma_e(G)$ of G is the minimum cardinality of a minimal edge dominating set of G. An edge dominating set I_e is called an independent edge dominating set if induced sub graph $\langle I_e \rangle$ is independent. A non trivial connected graph with no cut vertex is called a block. A block graph B(G) is a graph whose vertices are corresponding to the blocks of G and two vertices in B(G) are adjacent whenever the corresponding blocks contain a common cut vertex in G. In this paper we obtain certain bounds on $\gamma_{ib}(G)$ in terms of vertices, blocks and other parameters of G.

We need the following theorems to prove our later results.

Theorem A [3]: For any connected graph G, $\gamma(G) \leq \frac{p}{2}$.

Theorem B [4]: For any connected graph G, $\gamma(G) \leq n - \Delta(G)$.

Theorem C [4]: If G is a connected graph with p ≥ 3 vertices then $\gamma_c(G) = p \quad \varepsilon_T(G)$.

RESULTS

Theorem 1: For any connected graph G, $\gamma_{ib}(G) = 1$ if

- G is a non - separable graph. or
- G has exactly one cut vertex. or
- The block graph B(G) has exactly one cut vertex.

Proposition 1: For any path P_p,

$\gamma_{ib}(P_{3k-1}) = \gamma_{ib}(P_{3k}) = \gamma_{ib}(P_{3k+1}) = k$ where k = 1, 2, 3, ……….

Proposition 2: For any star $K_{1,n}$,

$\gamma_{ib}(K_{1,n}) = \gamma_i(K_{1,n}) = \gamma(K_{1,n}) = 1$.

Theorem 2: For any connected graph G, $\gamma_{ib}(G) \leq \gamma(G)$.

Proof: Let V = {v_1, v_2, v_3, ……., v_p} be the vertex set of G and let D = {v_1, v_2, ……., v_d} be the minimum dominating set of G. Let $A = E^I \cup M^I$ be the set of blocks of G where E^I is set of end blocks and M^I is set of non - end blocks in G. The corresponding block vertex set of A in B(G) is $H = E \cup M$ where E and M are corresponding block vertex sets of E^I *and* M^I respectively in B(G). Then the set $I = E_1 \cup M_1$ where $E_1 \subseteq E$ and $M_1 \subseteq M$ covers all the vertices of H – I and every $b_i \in I$ is at a distance at least 2 apart from other vertices of I. Also the induced sub graph $\langle E_1 \cup M_1 \rangle$ is disconnected. Then clearly, $|E_1 \cup M_1| = |I| = \gamma_{ib}(G)$. Now to give the relation with $\gamma(G)$ we consider the following cases.

Case 1: Suppose G is non - separable. Then B(G) is an isolated vertex and $\gamma_{ib}(G) = 1$. Since any non – separable graph has at least two vertices, clearly $\gamma(G) \geq \gamma_{ib}(G)$.

Case 2: Suppose G is separable. Then {$E_1 \cup M_1$} is an independent dominating set and since every $b_i \in I$ contains at least one vertex of D, clearly $\gamma_{ib}(G) \leq \gamma(G)$.

Theorem 3: For any connected graph G, $\gamma_{ib}(G) \leq \left\lfloor \frac{p}{2} \right\rfloor$.

Proof: By Theorem A and by Theorem 2 the result follows.

Theorem 4: For any connected (p, q) graph G with n blocks, $\gamma_{ib}(G) < \left\lfloor \frac{p+n}{2} \right\rfloor$.

Proof: Let G be a graph with p ≥ 2 vertices and A = {B1, B_2… B_n } be the set of blocks of G. Let H = {b_1, b_2, ……., b_n } be the corresponding block vertex set in B(G) respectively. Let I = {b_1, b_2, ……., b_i }, i < n be the minimum

independent block dominating set in B(G) and the minimum independent block domination number $\gamma_{ib}(G) = |I|$. We prove the result by induction on blocks of G. The inequality is clearly true for n = 1, 2 or 3 and the smallest graph for which the equality occurs is a block with two vertices. Assume the result is true for all graphs with n = k blocks. Then $\gamma_{ib}(G) \le \left\lfloor \frac{p+k}{3} \right\rfloor$. If G has k + 1 blocks and this (k + 1)th block is connected with any other block in G then at least one vertex b_t will be increased in B(G). If $N(b_i) \in I$ then $\gamma_{ib}(G) = |I|$. If $N(b_i) \notin I$ then $b_i \in I$ and $\gamma_{ib}(G) = |I| + 1 \le \left\lfloor \frac{p+k+1}{3} \right\rfloor$. Hence the result $\gamma_{ib}(G) \le \left\lfloor \frac{p+n}{3} \right\rfloor$ holds.

Theorem 5: For any connected (p, q) graph G, $\gamma_{ib}(G) \le \gamma_c(G)$. Equality holds for a graph G with exactly one cut vertex incident with blocks which are complete.

Proof: Let D_c be the γ_c- set of G. Then D_c must contain at least one vertex from each block of A = {$B_1, B_2, \ldots\ldots,$ B_n } in G. Let H = {$b_1, b_2, \ldots\ldots, b_n$ } be the corresponding block vertex set respectively in B(G). Let I = {b_i }, i < n be the minimum independent block dominating set in B(G). Since every $b_i \in I$, $1 \le i \le n$ has at least one vertex of D_c, clearly $|I| \le |D_c|$.

Hence $\gamma_{ib}(G) = |I| \le |D_c| = \gamma_c(G)$ gives $\gamma_{ib}(G) \le \gamma_c(G)$.

For equality, suppose the graph G has exactly one cut vertex incident with blocks which are complete. Clearly $\gamma_c(G) = 1$ and since B(G) is a complete graph the equality follows.

Theorem 6: For any connected graph G, $\left\lceil \frac{p-q}{1+\Delta(G)} \right\rceil \le \gamma_{ib}(G) \le p - \Delta(G)$.

Proof: For any connected graph $p - q \le 1$ and $\gamma_{ib}(G) \ge 1$. Also for any graph G, $1 + \Delta(G) \ge 2$. Hence $\left\lceil \frac{p-q}{1+\Delta(G)} \right\rceil \le \gamma_{ib}(G)$. For upper bound, by Theorem 5, $\gamma_{ib}(G) \le \gamma_c(G)$ and by Theorem B, $\gamma_c(G) \le p - \Delta(G)$. Hence $\gamma_{ib}(G) \le p - \Delta(G)$ and $\left\lceil \frac{p-q}{1+\Delta(G)} \right\rceil \le \gamma_{ib}(G) \le p - \Delta(G)$.

Theorem 7: For any tree T, $\gamma_{ib}(T) = \gamma_i^l(T)$.

Proof: Since every edge of T becomes a vertex of B(T) clearly, $\gamma_{ib}(T) = \gamma_i^l(T)$.

Theorem 8: For any graph G, $\gamma_{ib}(G) \le \alpha_0(G)$.

Proof: In [5], it is known that $\gamma(G) \le \alpha_0(G)$ and by Theorem 2, $\gamma_{ib}(G) \le \gamma(G)$. Hence the result holds.

Theorem 9: For any connected graph G with $p \ge 2$, $\gamma_{ib}(G) \le p - 1$.

Proof: Suppose G is non - separable. Then by Theorem 1, the inequality is true. Suppose G is separable. Let H = { $b_1, b_2, \ldots, b_n$ } be the block vertex set of B(G) and I = { b_i }, $1 \le i < n$ be the minimum independent block dominating set in B(G). Then maximum possible number of vertices in H is $n = p - 1$, obviously $\forall\, b_i \in I$, $|I| \le n = p - 1$. Hence $\gamma_{ib}(G) = |I| \le p - 1$ gives the result.

Theorem 10: For any connected graph G with $p \ge 3$, $\gamma_{ib}(G) \le p - |V_e|$.

Proof: By Theorem C and by Theorem 5 the result follows.

Theorem 11: For any connected graph G, $\gamma_{ib}(G) \leq p - \chi(G) + 1$. Equality holds for any complete graph with p ≥ 2 vertices.

Proof: From Theorem 2, $\gamma_{ib}(G) \leq \gamma(G)$.

In [1], D. Gernert has showed that $\gamma(G) + \chi(G) \leq n + 1$. When in fact,

$\gamma(G) \leq i(G) \leq \beta_0(G) \leq n - \chi(G) + 1$. Hence for our result one can easily verify the result.

Theorem 12: For any tree T, $\gamma_{ib}(T) \leq \frac{p-1}{2}$. Equality holds if and only if T is isomorphic to a spider.

Proof: Let $H_2 = \{ v_1, v_2, v_3, \ldots, v_j \}$ be the set of vertices in B(T) such that $H_2 \in N(H_1)$ where $H_1 = \{v_1, v_2, v_3, \ldots, v_i \}$ is the set of end vertices of B(T). Suppose $K \subseteq V[B(T)] - H_2$ and $K \notin N(H_1)$. Then $I = K_1 \cup H_2$ where $K_1 \subseteq K$ covers all the vertices in B(T). Further if sub graph $\{ K_1 \cup H_2 \}$ is independent then the set I forms a minimal independent dominating set of B(T). Clearly $|K_1 \cup H_2| \leq \frac{p-1}{2}$. It follows that $\gamma_{ib}(T) \leq \frac{p-1}{2}$. For equality we consider the following.

For necessary condition consider $\gamma_{ib}(T) \leq \frac{p-1}{2}$.

Since number of vertices in H is p – 1, $\gamma_{ib}(T) \leq \frac{p-1}{2} = \frac{H}{2}$. Since I is minimal independent dominating set, for any $v_i \in I, N(v_i) \notin I$. It follows that all $v_i \in H_1, N(v_i) \in H_2$ and $|H_1| = |H_2|$. So $H_1 \cup H_2$ is V [B(T)] and the tree T is a spider.

For sufficient condition, Suppose the tree T is isomorphic to a spider. Then $|I| = |H_1| = \frac{q}{2}$. Since for any tree T, q = p – 1 it follows that $\gamma_{ib}(T) = \frac{p-1}{2}$.

Theorem 13: For any tree T, $\gamma_{ib}(T) \leq \left\lceil \frac{q-m}{2} \right\rceil + 1$ where m is number of end blocks of T.

Proof: Let $A = \{B_1, B_2, \ldots, B_n \}$ be the set of blocks of T and let $E^I = \{ B_1, B_2, \ldots, B_m \}$ be the set of all end blocks in T with $|E^I| = m$. The corresponding block vertex sets are $H = \{ b_1, b_2, \ldots, b_n \}$ and $E = \{ b_1, b_2, \ldots, b_m \}$ respectively in B(T). Without loss of generality since $|H| = |A|$, let $I = F \cup K$ where $F \subseteq E$ and $K \subseteq H - E$ such that $K \notin N(E)$ be the minimal set of vertices that covers all the vertices in B(T). By the above condition clearly the vertices of < I > are independent. Then I is minimal independent dominating set of B(T). Suppose ∃ at least one vertex v in T with deg v ≥ 3 incident with end blocks. Then in B(T), the edges incident to v form a complete graph as a block in B(T).

Suppose T has more than one vertex with deg v ≥ 3. Then B(T) has more than one block which is complete with p ≥ 3 vertices. Let $H_3 = \{b_1, b_2, \ldots, b_k \}$ be the vertex set and each vertex b_k belongs to each block of B(T) which is complete. Let $H_4 = \{b_1, b_2, \ldots, b_t \}$ be the vertex set such that any b_t does not belonging to end blocks of B(T). Hence $\forall e_t \in E(B(T))$ are not joining any two vertices in $\{ H_3 \cup H_4 \}$. Clearly $\left\lceil \frac{q-m}{2} \right\rceil \geq |H_3 \cup H_4| - 1$, which gives $|H_3 \cup H_4| \leq \left\lceil \frac{q-m}{2} \right\rceil + 1$.

CONCLUSIONS

We studied the property of our concept by applying to some standard graphs. We also established the results by comparing our concept to other different dominating parameters. Also compared to chromatic number of general graph, so that to develop further results related to the NP – problems.

REFERENCES

1. D. Gernert, Inequalities between the domination number and the chromatic number of a graph, Discrete Math. 76: 151 – 153, 1989.
2. Harary F, Graph Theory, Addison – Wesley Reading Mass (1974).
3. S. T. Hedetniemi & R. C. Laskar, Connected domination in graphs. In B. Bollobas, editor, Graph Theory and combinatorics, Pages 209 – 218, Academic press, London, 1984.
4. O. Ore, Theory of graphs, Amer Math. Soc. Colloq. publ. 38, providence, (1962).
5. Teresa. W. Haynes, Stephen T. Hedetniemi, Peter J. Slater, Fundamentals of domination in graphs, page 58, 1998.